MEMC-5173
Intelligent Tools for Engineering Applications

Assignment #2 reference solution

Q1. (Question 2.28 from Book 1)

Use the extension principle.

$$\mu_A(x) = \frac{x+1}{x+2} \quad \text{for } X = [0, 8]$$

$$y = x+1 \quad \Rightarrow \quad x = y-1$$

Hence,

$$\mu_B(y) = \frac{y-1+1}{y-1+2} \quad \text{for } Y = [0+1,\ 8+1]$$

or

$$\mu_B(y) = \frac{y}{y+1} \quad \text{for } Y = [1,\ 9]$$

Q2. (Question 2.20 from Book 1)

Redo Example 2.16, and use the results for analysis.
(Part of the solutions)

Consider a fuzzy set A, with membership function μ_A, in a universe of discourse X and a fuzzy set B, with membership function μ_B, in a universe of discourse Y. The fuzzy implication $A \rightarrow B$ is a fuzzy relation in the Cartesian product space $X \times Y$. In classical propositional calculus, we can show by writing truth tables that the implication $A \rightarrow B$ is equivalent to "(Not A) Or B", which is written as

$$A \rightarrow B \equiv \overline{A} \vee B. \qquad \text{(i)}$$

This is also equivalent to

$$A \rightarrow B \equiv (A \wedge B) \vee \overline{A} \qquad \text{(ii)}$$

because

$$(A \wedge B) \vee \overline{A} = (A \vee \overline{A}) \wedge (B \vee \overline{A}) = X \wedge (B \vee \overline{A}) = B \vee \overline{A} = \overline{A} \vee B.$$

Note that we used the distributivity of $\vee$ over $\wedge$ and the commutativity of $\vee$; where -, $\wedge$, and $\vee$ represent (classical) logic operations "not", "and", and "or", respectively. The fuzzy implication can be viewed as a generalized crisp operation where the A and B represent fuzzy propositions, and the operations -, $\wedge$, and $\vee$ represent fuzzy complement, intersection, and union, respectively.

Next, note that $A \rightarrow B$ can be also interpreted as a local implication; that is, $A \rightarrow B$ also means $\overline{A} \rightarrow \overline{B}$. In other words, $A \rightarrow B$ would also mean $B \rightarrow A$ (because $A \rightarrow B$ also implies $\overline{B} \rightarrow \overline{A}$; and $\overline{A} \rightarrow \overline{B}$ also implies $B \rightarrow A$). This is not globally true, however, and is clearly not true for crisp binary logic as the truth tables show that $A \rightarrow B$ is not identical to $B \rightarrow A$. But, in a local sense, we can write

$$A \rightarrow B \equiv A \wedge B. \qquad \text{(iii)}$$

This interpretation (iii) of implication can be thought of as a stronger form of (ii) in view of the fact that (ii) represents "(A And B) Or (Not A)" and (iii) represents "A And B".

It should be clear that the classical (weaker and global) definitions of implication, given by (i) and (ii), mean that "A entails B". But, the local (stronger) definition of implication, given by (iii), means that "A coupled with B". Furthermore, in the relations (i), (ii), and (iii) the sets or propositions A and B are defined in different universes X and Y. Hence, several methods of fuzzy implication, as derived from (i), (ii), and (iii) are given below:

(a) Larsen implication

$$\mu_{A\to B}(x,y)=\mu_A(x)\mu_B(y) \qquad \forall x\in X, \forall y\in Y$$

(b) Mamdani implication

$$\mu_{A\to B}(x,y)=\min\{\mu_A(x),\mu_B(y)\} \qquad \forall x\in X, \forall y\in Y$$

(c) Zadeh implication

$$\mu_{A\to B}(x,y)=\max[\min\{\mu_A(x),\mu_B(y)\},1-\mu_A(x)] \qquad \forall x\in X, \forall y\in Y$$

(d) Dienes-Rescher implication

$$\mu_{A\to B}(x,y)=\max\{1-\mu_A(x),\mu_B(y)\} \qquad \forall x\in X, \forall y\in Y$$

(e) Lukasiewicz implication (Bounded sum)

$$\mu_{A\to B}(x,y)=\min\{1,1-\mu_A(x)+\mu_B(y)\} \qquad \forall x\in X, \forall y\in Y$$

where $\mu_{A\to B}(x,y)$ is the membership function of the relation, which represents fuzzy implication $A \to B$. These results are summarized in Table 2.

Table S2.20 Some Interpretations of Fuzzy Implication.

Representation of $A \to B$	Meaning	Comments	Examples
$\overline{A} \vee B$	(Not A) Or B	• Same as in crisp binary logic • A weaker (global) implication, meaning "A entails B"	• Dienes-Rescher Implication $\max\{1-\mu_A(x),\mu_B(y)\}$ • Lukasiewicz Implication $\min\{1,1-\mu_A(x)+\mu_B(y)\}$
$(A \wedge B) \vee \overline{A}$	(A And B) Or (Not A)	same as above	• Zadeh Implication $\max[\min\{\mu_A(x),\mu_B(y)\},1-\mu_A(x)]$
$A \wedge B$	A And B	• A stronger (local) implication, meaning "A is coupled with B"	• Mamdani Implication $\min\{\mu_A(x),\mu_B(y)\}$ • Larsen Implication $\mu_A(x)\mu_B(y)$

Numerical Example:

Consider the membership functions of fuzzy set A and B as shown in Figure S2.20, and expressed below:

$\mu_A(x) = \max\{0, \frac{10x-3}{2}\}$ $\quad 0.3 \le x \le 0.5$ $\qquad \mu_B(y) = \max\{0, \frac{10y-3}{2}\}$

$0.3 \le y \le 0.5$

$= \max\{0, \frac{7-10x}{2}\}$ $\quad 0.5 < x \le 0.7$ $\qquad = \max\{0, \frac{7-10y}{2}\}$

$0.5 < y \le 0.7$

$= 0$ otherwise $\qquad = 0$

otherwise

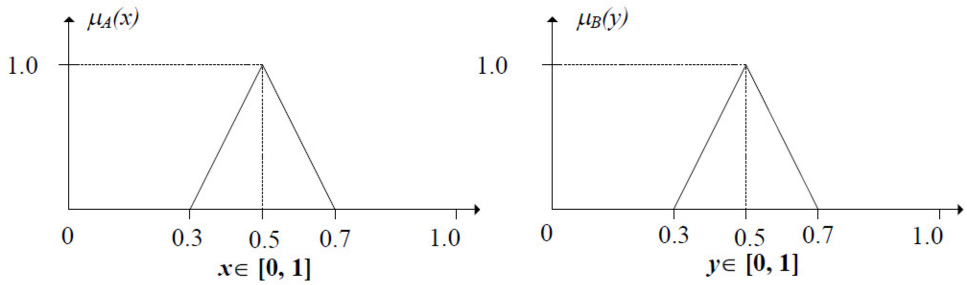


Figure S2.20 Membership functions of fuzzy sets A and B.

The resulting expressions for the membership functions that represent the candidate implication are given below, and illustrated in Figure 2.11.

(a) Larsen implication (product or dot operation)

$$\mu_{A\to B}(x,y) = \begin{cases} \frac{(10x-3)(10y-3)}{4} & \text{if } 0.3 \le x \le 0.5 \text{ AND } 0.3 \le y \le 0.5 \\ \frac{(10x-3)(7-10y)}{4} & \text{if } 0.3 \le x \le 0.5 \text{ AND } 0.5 < y \le 0.7 \\ \frac{(7-10x)(10y-3)}{4} & \text{if } 0.5 < x \le 0.7 \text{ AND } 0.3 \le y \le 0.5 \\ \frac{(7-10x)(7-10y)}{4} & \text{if } 0.5 < x \le 0.7 \text{ AND } 0.5 < y \le 0.7 \\ 0 & \text{if } \quad \text{otherwise} \end{cases}$$

(b) Mamdani implication (min operation)

$$\mu_{A\to B}(x,y) = \begin{cases} \min[\frac{10x-3}{2}, \frac{10y-3}{2}] & \text{if } 0.3 \le x \le 0.5 \text{ AND } 0.3 \le y \le 0.5 \\ \min[\frac{10x-3}{2}, \frac{7-10y}{2}] & \text{if } 0.3 \le x \le 0.5 \text{ AND } 0.5 < y \le 0.7 \\ \min[\frac{7-10x}{2}, \frac{10y-3}{2}] & \text{if } 0.5 < x \le 0.7 \text{ AND } 0.3 \le y \le 0.5 \\ \min[\frac{7-10x}{2}, \frac{7-10y}{2}] & \text{if } 0.5 < x \le 0.7 \text{ AND } 0.5 < y \le 0.7 \\ 0 & \text{if } \quad \text{otherwise} \end{cases}$$

(c) Zadeh implication

$\mu_{A\to B}(x,y) =$

$$\begin{cases} \max[\min\{\frac{10x-3}{2},\frac{10y-3}{2}\},\frac{5-10x}{2}] & \text{if} \quad 0.3 \le x \le 0.5 \text{ AND } 0.3 \le y \le 0.5 \\ \max[\min\{\frac{10x-3}{2},\frac{7-10y}{2}\},\frac{5-10x}{2}] & \text{if} \quad 0.3 \le x \le 0.5 \text{ AND } 0.5 < y \le 0.7 \\ \frac{5-10x}{2} & \text{if} \quad 0.3 \le x < 0.5 \text{ AND } (0 \le y < 0.3 \text{ OR } 0.7 < y \le 1) \\ \max[\min\{\frac{7-10x}{2},\frac{10y-3}{2}\},\frac{10x-3}{2}] & \text{if} \quad 0.5 < x \le 0.7 \text{ AND } 0.3 \le y \le 0.5 \\ \max[\min\{\frac{7-10x}{2},\frac{7-10y}{2}\},\frac{10x-3}{2}] & \text{if} \quad 0.5 < x \le 0.7 \text{ AND } 0.5 < y \le 0.7 \\ \frac{10x-5}{2} & \text{if} \quad 0.5 < x \le 0.7 \text{ AND } (0 \le y < 0.3 \text{ OR } 0.7 < y \le 1) \\ 1 & \text{if} \quad \text{otherwise} \end{cases}$$

(d) Dienes-Rescher implication

$\mu_{A\to B}(x,y) =$

$$\begin{cases} \max[\frac{5-10x}{2},\frac{10y-3}{2}] & \text{if} \quad 0.3 \le x \le 0.5 \text{ AND } 0.3 \le y \le 0.5 \\ \max[\frac{5-10x}{2},\frac{7-10y}{2}] & \text{if} \quad 0.3 \le x \le 0.5 \text{ AND } 0.5 < y \le 0.7 \\ \frac{5-10x}{2} & \text{if} \quad 0.3 \le x < 0.5 \text{ AND } (0 \le y < 0.3 \text{ OR } 0.7 < y \le 1) \\ \max[\frac{10x-5}{2},\frac{10y-3}{2}] & \text{if} \quad 0.5 < x \le 0.7 \text{ AND } 0.3 \le y \le 0.5 \\ \max[\frac{10x-5}{2},\frac{7-10y}{2}] & \text{if} \quad 0.5 < x \le 0.7 \text{ AND } 0.5 < y \le 0.7 \\ \frac{10x-5}{2} & \text{if} \quad 0.5 < x \le 0.7 \text{ AND } (0 \le y < 0.3 \text{ AND } 0.7 < y \le 1) \\ 1 & \text{if} \quad \text{otherwise} \end{cases}$$

(e) Lukasiewicz implication

$$\mu_{A\to B}(x,y)=\begin{cases}\min[1,1-5x+5y] & \text{if} \quad 0.3\le x\le 0.5 \text{ AND } 0.3\le y\le 0.5\\ \min[1,6-5x-5y] & \text{if} \quad 0.3\le x\le 0.5 \text{ AND } 0.5< y\le 0.7\\ \min[1,\frac{5-10x}{2}] & \text{if} \quad 0.3\le x< 0.5 \text{ AND } (0\le y<0.3 \text{ AND } 0.7<y\le 1)\\ \min[1,5x+5y-4] & \text{if} \quad 0.5< x\le 0.7 \text{ AND } 0.3\le y\le 0.5\\ \min[1,1+5x-5y] & \text{if} \quad 0.5< x\le 0.7 \text{ AND } 0.5< y\le 0.7\\ \min[1,\frac{10x-5}{2}] & \text{if} \quad 0.5< x\le 0.7 \text{ AND } (0\le y<0.3 \text{ OR } 0.7<y\le 1)\\ \min[1,\frac{10y+1}{2}] & \text{if} \quad (0\le x<0.3 \text{ OR } 0.7<x\le 1) \text{ OR } 0.3\le y\le 0.5)\\ \min[1,\frac{9-10y}{2}] & \text{if} \quad (0\le x<0.3 \text{ OR } 0.7<x\le 1) \text{ OR } 0.5< y\le 0.7)\\ 1 & \text{if} \quad \text{otherwise}\end{cases}$$

Among these fuzzy implication operations, one can find a relationship of ordering, as follows:
Since $0\le 1-\mu_A(x)\le 1$ and $0\le \mu_B(y)\le 1$, we have

$$max[1-\mu_A(x),\mu_B(y)]\le 1-\mu_A(x)+\mu_B(y) \text{ and } max[1-\mu_A(x),\mu_B(y)]\le 1.$$

Hence,

$$max[1-\mu_A(x),\mu_B(y)]\le min[1,1-\mu_A(x)+\mu_B(y)].$$

Moreover, since $min[\mu_A(x),\mu_B(y)]\le \mu_B(y)$, we have

$$max[min\{\mu_A(x),\mu_B(y)\},1-\mu_A(x)]\le max[\mu_B(y),1-\mu_A(x)].$$

Also, since $max(p,a)\ge p$, we have

$$max[min\{\mu_A(x),\mu_B(y)\},1-\mu_A(x)]\ge min[\mu_A(x),\mu_B(y)].$$

Furthermore, since "min" is the largest T-norm and the "product" is another T-norm, we have

$$min[\mu_A(x),\mu_B(y)]\ge \mu_A(x)\mu_B(y).$$

It follows that

Larsen $\subseteq$ *Mamdani* $\subseteq$ *Zadeh* $\subseteq$ *Dienes-Rescher* $\subseteq$ *Lukasiewicz*.

From the relationship, Larsen implication is the most local (strongest) one among them.

Mamdani implication is the most widely used implication in fuzzy systems and fuzzy control because it is simpler to use, robust, and often provides good results. A similar comment can be made for Larsen implication. Unlike the remaining implication operations, Mamdani and Larsen implications are based on the concept that fuzzy IF-THEN rules are local. For example, when we say "IF temperature is high THEN failure potential is high," we deal with a local situation. It should not be used to ascertain other local situations such as "temperature is high" and "temperature is medium". Furthermore, according to Mamdani and Larsen implications, this also implies "IF temperature is not high THEN failure potential is not high". This is not generally true, however. Consequently, different implications might be needed to cope with diverse applications.

Q3. (Example 3.4 in Book 2)

On page 52, determine $\Re_1 \circ \Re_2$ in the degrees of relevance between (1, *a*), (1, *b*), (3, *a*), (3, *b*), by max-min and max-product compositions, respectively.

1) Using max-min composition:

$$\mu_{\Re_1 \circ \Re_2}(1,\ a) = \max[0.1 \wedge 0.9,\ 0.3 \wedge 0.2,\ 0.5 \wedge 0.5,\ 0.7 \wedge 0.7]$$
$$= \max[0.1,\ 0.2,\ 0.5,\ 0.7] = 0.7$$

$$\mu_{\Re_1 \circ \Re_2}(1,\ b) = \max[0.1 \wedge 0.1,\ 0.3 \wedge 0.3,\ 0.5 \wedge 0.6,\ 0.7 \wedge 0.2]$$
$$= \max[0.1,\ 0.3,\ 0.5,\ 0.4] = 0.5$$

$$\mu_{\Re_1 \circ \Re_2}(3,\ a) = \max[0.6 \wedge 0.9,\ 0.8 \wedge 0.2,\ 0.3 \wedge 0.5,\ 0.2 \wedge 0.7]$$
$$= \max[0.6,\ 0.2,\ 0.3,\ 0.2] = 0.6$$

$$\mu_{\Re_1 \circ \Re_2}(3,\ a) = \max[0.6 \wedge 0.1,\ 0.8 \wedge 0.3,\ 0.3 \wedge 0.6,\ 0.2 \wedge 0.2]$$
$$= \max[0.1,\ 0.3,\ 0.3,\ 0.2] = 0.3$$

2) Using max-product composition:

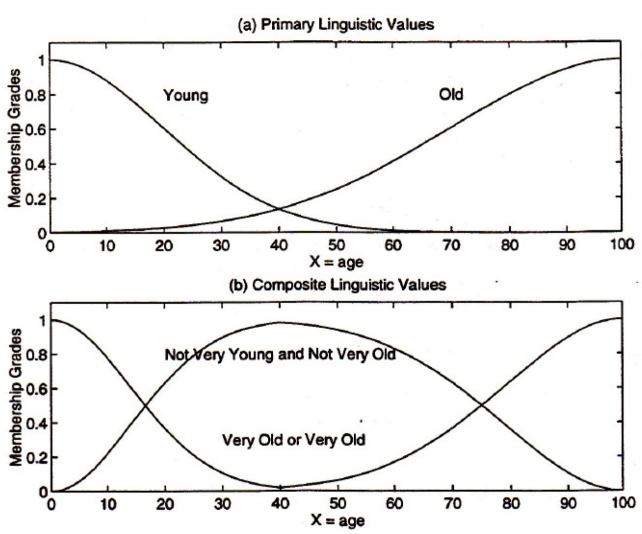
$$\mu_{\Re_1 \circ \Re_2}(1,\ a) = \max[0.1 \times 0.9,\ 0.3 \times 0.2,\ 0.5 \times 0.5,\ 0.7 \times 0.7]$$
$$= \max[0.09,\ 0.06,\ 0.25,\ 0.49] = 0.49$$

$$\mu_{\Re_1 \circ \Re_2}(1,\ b) = \max[0.1 \times 0.1,\ 0.3 \times 0.3,\ 0.5 \times 0.6,\ 0.7 \times 0.2]$$
$$= \max[0.01,\ 0.09,\ 0.30,\ 0.14] = 0.30$$

$$\mu_{\Re_1 \circ \Re_2}(3,\ a) = \max[0.6 \times 0.9,\ 0.8 \times 0.2,\ 0.3 \times 0.5,\ 0.2 \times 0.7]$$
$$= \max[0.54,\ 0.16,\ 0.15,\ 0.14] = 0.54$$

$$\mu_{\Re_1 \circ \Re_2}(3,\ a) = \max[0.6 \times 0.1,\ 0.8 \times 0.3,\ 0.3 \times 0.6,\ 0.2 \times 0.2]$$
$$= \max[0.06,\ 0.24,\ 0.18,\ 0.05] = 0.24$$

Q4. (Question 3.6 in Book 2)



Q 5. (Question 3.10 in Book 2)

Just use a truth table to verify Equation (3.19).

We shall prove that the right hand sides of Equations (3.19) to (3.22) reduce to $A \to B \equiv \neg A \cup B$ when when A and B assume the values in two-value logic.

(a) The right hand side of Equation (3.19) is already in the format we want.

(b)

$$\begin{aligned} &\text{Right hand side of Equation (3.20)} \\ &= \neg A \cup (A \cap B) \\ &= (\neg A \cup A) \cap (\neg A \cup B) \\ &= U \cap (\neg A \cup B) \\ &= \neg A \cup B. \end{aligned}$$

(c)

$$\begin{aligned} &\text{Right hand side of Equation (3.21)} \\ &= (\neg A \cap \neg B) \cup B \\ &= (\neg A \cup B) \cap (\neg B \cup B) \\ &= (\neg A \cup B) \cap U \\ &= \neg A \cup B. \end{aligned}$$

(d) We can construct a truth table as follows:

$\mu_A(x)$	$\mu_B(y)$	$\mu_R(x,y)$	$\mu_{\neg A}(x)$	$\mu_{\neg A \cup B}(x,y)$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Therefore $\mu_R(x,y) = \mu_{\neg A \cup B}(x,y)$, which implies $R = \neg A \cup B$.